Residue currents of coherent sheaves via superconnections

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Contents

The topological space $\Omega_c^{p,q}(X)$

- Let X be an n-dimensional complex manifold.
- Let $\Omega_c^{p,q}(X)$ be the space of compactly supported (p,q)-forms on X, equipped with the usual topology given as follows:
- We say $\omega_n \to \omega$ if for any coordinate chart U and any multi-index α we have

$$||\partial^\alpha \omega_n - \partial^\alpha \omega|| \to 0$$

on U.

Review of currents

- A (p,q)-*current* on X is a continuous linear map from $\Omega_c^{n-p,n-q}(X)$ to $\mathbb{C}$.
- We denote the set of (p,q)-currents on X by $\mathcal{D}^{p,q}(X)$.
- Examples of currents:
  - If $\omega = \sum_{|I|=p,|J|=q} f_{IJ} dz^I \wedge d\bar{z}^J$ where f_{IJ} is a *locally integrable* function on X, then
  $$\omega(\theta) := \int_X \omega \wedge \theta, \ \forall \theta \in \Omega_c^{n-p,n-q}(X)$$
  defines a (p,q)-current on X.
  - If Z is a subvariety of codimension p, then
  $$[Z](\theta) := \int_Z \theta, \ \forall \theta \in \Omega_c^{n-p,n-p}(X)$$
  defines a (p,p)-current on X.

New currents from old ones

- We can extend operations on differential forms to operations on currents by *duality*.
- Let $T \in \mathcal{D}^{p,q}(X)$ be a (p,q)-current on X. We can define a $(p,q+1)$-current $\bar{\partial}T$ as
$$\bar{\partial}T(\theta) := (-1)^{p+q+1}T(\bar{\partial}\theta),\ \forall \theta \in \Omega_c^{n-p,n-q-1}(X).$$
We can define ∂T in a similar way.
- It is compatible with the $\bar{\partial}$-operation on $\Omega^{\bullet,\bullet}(X)$ because for $\omega \in \Omega^{p,q}(X)$ considered as a current as before, we have
$$0 = \int_X \bar{\partial}(\omega \wedge \theta) = \int_X (\bar{\partial}\omega) \wedge \theta + (-1)^{p+q} \int_X \omega \wedge \bar{\partial}\theta.$$

New currents from old ones (cont'd)

- For a $\omega \in \Omega^{s,t}(X)$ and $T \in \mathcal{D}^{p,q}(X)$, we can define a current $\omega \wedge T \in \mathcal{D}^{p+s,q+t}(X)$ as
$$(\omega \wedge T)(\theta) := (-1)^{(s+t)(p+q)} T(\omega \wedge \theta),\ \forall \theta \in \Omega_c^{n-p-s,n-q-t}(X).$$
We can define $T \wedge \omega$ in a similar way.
- In general we *cannot* define the wedge product of two currents.
- We have an inclusion of cochain complexes $(\Omega^{\bullet,\bullet}(X), \bar{\partial}) \hookrightarrow (\mathcal{D}^{\bullet,\bullet}(X), \bar{\partial})$.
- Elliptic regularity theory: The above inclusion is a quasi-isomorphism, i.e. we can compute the Dolbeault cohomology of X by currents.

Holomorphic function and Poincaré-Lelong formula

- Let f be a generically nonvanishing holomorphic function on X.
- Let Z_f be the zero locus of f hence we have a $(1,1)$-current $[Z_f]$.
- $\log|f|^2$ is locally integrable.
- Hence we can define a $(1,1)$-current $\bar{\partial}\partial\log|f|^2$.

Theorem (Poincaré-Lelong formula)

We have an equality of currents

$$\frac{1}{2\pi i}\bar{\partial}\partial\log|f|^2 = [Z_f].$$

More on Poincaré-Lelong formula

- We know that $\bar{\partial}\partial = -\partial\bar{\partial}$.
- For a holomorphic function f, we have $\bar{\partial} f = 0$, $\partial \bar{f} = 0$, hence

$$\bar{\partial}\partial f = 0 \text{ and } \bar{\partial}\partial \bar{f} = 0.$$

- Conceptually we have

$$\begin{aligned}\bar{\partial}\partial \log|f|^2 &= \bar{\partial}\partial(\log f + \log \bar{f}) = \bar{\partial}\left(\frac{\partial f}{f} + \frac{\partial \bar{f}}{\bar{f}}\right)\\ &= \bar{\partial}\left(\frac{1}{f}\right) \wedge \partial f + 0 = \bar{\partial}\left(\frac{1}{f}\right) \wedge df.\end{aligned}$$

Problem

In general $\frac{1}{f}$ is not locally integrable, so $\frac{1}{f}$ and $\bar{\partial}(\frac{1}{f})$ are not currents on X in the naive sense.

Residue current of a function

- [Dolbeault, 1971] and [Herrera and Lieberman, 1971] solved this problem by defining the **principle value current** $\frac{1}{f}$ and the **residue current** $\bar{\partial}(\frac{1}{f})$ as
$$(\frac{1}{f})(\omega) := \lim_{\epsilon\to 0}\int_{|f|>\epsilon}\frac{\omega}{f}, \text{ and } \bar{\partial}(\frac{1}{f})(\psi) := \lim_{\epsilon\to 0}\int_{|f|=\epsilon}\frac{\psi}{f}$$
for a testing $2n$-form ω and $(2n-1)$-form ψ.
- $\bar{\partial}(\frac{1}{f})$ is a well-defined $(0,1)$-current, which we also denote by R_f.

A baby example

- Let $X = \mathbb{C}$ and $f = z$.
- If we write the testing $(1,0)$-form θ as $\theta = s(z)dz$, then a polar coordinate computation shows $\frac{1}{2\pi i}\bar{\partial}(\frac{1}{z})(\theta) = s(0)$.
- For a testing function $s(z)$ we have
$$\frac{1}{2\pi i}\bar{\partial}(\frac{1}{z}) \wedge (dz)(s(z)) = \frac{1}{2\pi i}\bar{\partial}(\frac{1}{z})(s(z) \wedge dz) = s(0).$$
- On the other hand $Z_f = \{0\}$
- We checked $\bar{\partial}(\frac{1}{z}) \wedge (dz) = \bar{\partial}\partial \log |z|^2$ and Poincaré-Lelong formula by hand.

Duality principle

Proposition (Duality principle)

A holomorphic function g *on* X *is a multiple of* f *if and only if* $g\bar{\partial}(\frac{1}{f}) = 0$ *as a current.*

- In the case $X = \mathbb{C}$ and $f = z$, the duality principle says: $g(z)$ is a multiple of z if and only if $g(0) = 0$.
- The proof of the results in [Dolbeault, 1971] and [Herrera and Lieberman, 1971] in the general case depends on *Hironaka's desingularization theorem*.

Residue current of a collection of functions

- Let $f = (f_1, \ldots, f_m)$ be a collection of holomorphic functions.
- A path $\epsilon(t) = (\epsilon_1(t), \ldots, \epsilon_m(t))$ in $\mathbb{C}^m$ is called *admissible* if
$$\lim_{t\to 0} \epsilon_m(t) = 0, \text{ and } \lim_{t\to 0} \frac{\epsilon_j(t)}{(\epsilon_{j+1}(t))^q} = 0, \ j = 1, \ldots, m-1,$$
for any positive integer q.
- [Coleff and Herrera, 1978]: We can define the residue current R_f of f as
$$R_f(\psi) := \lim_{t\to 0} \int_{|f_1|=\epsilon_1(t), \ldots, |f_m|=\epsilon_m(t)} \frac{\psi}{f_1 \cdot \ldots \cdot f_m}$$
where $\epsilon(t) = (\epsilon_1(t), \ldots, \epsilon_m(t))$ is an admissible path and ψ is a test $(2n-m)$-form.
- R_f is a well-defined $(0, m)$-current. Heuristically we can consider it as the (noncommutative) wedge product $R_f = \bar{\partial}(\frac{1}{f_1}) \wedge \ldots \wedge \bar{\partial}(\frac{1}{f_m})$.

Review: currents valued in vector bundles

Let E be a C^∞-vector bundle on X with dual bundle E^*.

- A (p,q)-current valued in E is a continuous linear map from $\Omega_c^{n-p,n-q}(X,E^*)$ to $\mathbb{C}$.
- A (p,q)-current valued in $\mathrm{End}(E)$ is a continuous linear map from $\Omega_c^{n-p,n-q}(X,\mathrm{End}(E))$ to $\mathbb{C}$.
- We can define wedge products, differential operators, traces, etc. on bundle-valued current as before.

Complexes of holomorphic vector bundles and minimal right inverse

[Andersson and Wulcan, 2007]

- Let

$$\xi : 0 \to E_{-N} \xrightarrow{\phi_{-N}} E_{-N+1} \xrightarrow{\phi_{-N+1}} \dots \xrightarrow{\phi_{-1}} E_0 \to 0$$

  be a bounded complex of holomorphic vector bundles.
- We equip each E_i with a Hermitian metric.
- For each $i = -1, \dots, -N$, let $\sigma_i : E_{i+1} \to E_i$ be the **minimal right inverse** of ϕ_i.
- Minimal right inverse is defined by the following properties:

$$\phi_i \sigma_i|_{\mathrm{im}\phi_i} = \mathrm{id}_{\mathrm{im}\phi_i},\ \sigma_i|_{(\mathrm{im}\phi_i)^\perp} = 0, \text{ and } \mathrm{im}\sigma_i \perp \ker \phi_i \Rightarrow \sigma_{i-1}\sigma_i = 0.$$

- Minimal right inverse exists.

Minimal right inverse: an example

- $\underline{\mathbb{C}}^m$ the m-dimensional trivial vector bundle on X equipped with the standard Hermitian metric.
- A map $\phi : \underline{\mathbb{C}}^m \to \underline{\mathbb{C}}$ is given by $\phi = (f_1, \ldots, f_m)$ where $f_1, \ldots, f_m$ are C^∞-functions on X.
- If all f_i's are identically 0 on X, then the maximal rank of im ϕ is 0, hence $Z = \emptyset$ and $\sigma \equiv 0$.
- If some f_i's are not identically 0 on X, then the maximal rank of im ϕ is 1, hence $Z = \{x \in X | f_1(x) = \ldots = f_m(x) = 0\}$ and

$$\sigma(x) = \begin{cases} 0 & x \in Z \\ \frac{1}{\sum_{i=1}^m |f_i|^2} \begin{pmatrix} \overline{f_1} \\ \ldots \\ \overline{f_m} \end{pmatrix} & x \in X \backslash Z. \end{cases}$$

Minimal right inverse: properties

- σ_i could be singular, i.e., it could go to ∞.
- Let Z be the union of all singular locus of the σ_i's. Z has positive codimension in X.
- We are mostly interested in the case that ξ is acyclic on $X \backslash Z$.
- $\bar{\partial}\sigma_i$ may be nonzero, even when restricted to $X \backslash Z$.
- Notation

$$E^{\bullet} := \bigoplus_{i=-N}^{-1} E_i,$$

$$\phi := \phi_{-N} + \phi_{-N+1} + \ldots + \phi_{-1},$$

$$\sigma := \sigma_{-N} + \sigma_{-N+1} + \ldots + \sigma_{-1}.$$

The preimage problem

- If ξ is acyclic, then we can check $\sigma\phi + \phi\sigma = \mathrm{id}_{E^\bullet}$.

Question

If ξ is acyclic, then for $e \in E_0$ a holomorphic section, can we find a holomorphic element $x \in E_{-1}$ such that

$$\phi x = e?$$

- Naive answer: $x = \sigma e$, hence

$$\phi x = \phi\sigma e = (\sigma\phi + \phi\sigma)e = e.$$

- Problem: x is not holomorphic.

The construction of u

- On $X\backslash Z$ we define the $\mathrm{End}(E^\bullet)$-valued form

$$u := \sigma(\mathrm{id}_{E^\bullet} - \bar\partial\sigma)^{-1} = \sigma + \sigma(\bar\partial\sigma) + \sigma(\bar\partial\sigma)^2 + \dots$$

- If ξ is acyclic, then for $e \in E_0$ a holomorphic section, the equation

$$(\phi - \bar\partial)x = e$$

  has a solution in $\oplus_{i=-N}^{-1}\Omega^{0,-i-1}(X, E_i)$ given by ue.
- $([\phi, u] - \bar\partial u)e = e$
- The $\bar\partial$-operator is locally exact.
- Locally on X we can find $\tilde{x} \in E_{-1}$ such that $\tilde{x}$ is holomorphic and $\phi\tilde{x} = e$.
- u plays the role of $\frac{1}{f}$ before.

Almost semi-meromorphic and pseudomeromorphic currents

We follow [Andersson and Wulcan, 2010, Andersson and Wulcan, 2018].

- We can define **almost semi-meromorphic** currents on X, which generalize principal value currents $[\frac{1}{f}]$.
- We can define **pseudomeromorphic** currents on X, which generalize residue currents $\bar{\partial}(\frac{1}{f_1}) \wedge \ldots \wedge \bar{\partial}(\frac{1}{f_m})$.
- We can extend u to a $\mathrm{End}(E^\bullet)$-valued, almost semi-meromorphic current U on X.
- We can define a current $[\phi, U] - \bar{\partial}U$, which is a $\mathrm{End}(E^\bullet)$-valued, pseudomeromorphic current U on X.

Residue current of ξ

- The **residue current** R_ξ of the cochain complex ξ is an $\mathrm{End}(E^\bullet)$-valued, pseudomeromorphic current defined by
$$R_\xi := \mathrm{id}_{E^\bullet} - [\phi, U] + \bar{\partial} U.$$
- R_ξ measures how the cochain complex ξ fails to be acyclic.
- It is easy to see that when ξ is the complex $\underline{\mathbb{C}} \xrightarrow{f} \underline{\mathbb{C}}$, then R_ξ reduce to R_f.

Some notations

- Let $R_{\xi}^{i\to j}$ denote the component of R_{ξ} that maps E_i to E_j.
- For a coherent sheaf $\mathcal{F}$, we define its **cycle** as the current

$$[\mathcal{F}] := \sum_i m_i[Z_i]$$

  where Z_i is the irreducible component of the support of $\mathcal{F}$ and m_i is the geometric multiplicity of Z_i in $\mathcal{F}$.
- We say $\mathcal{F}$ has pure codimension p if supp$\mathcal{F}$ has pure codimension p.

Duality principle and generalized Poincaré-Lelong

Theorem (Duality principle, [Andersson and Wulcan, 2007])

If ξ is acyclic on $X \backslash Z$, then for a holomorphic section e of E^0, $R_\xi e = 0$ if and only if e can be locally written as ϕx where x is a holomorphic section of E_{-1}.

Theorem (Generalized Poincaré-Lelong formula, [Lärkäng and Wulcan, 2021])

If ξ is a resolution of a coherent sheaf $\mathcal{F}$ with pure codimension p. Then

$$\frac{1}{(2\pi i)^p p!} tr(D\phi_{-1}) \dots (D\phi_{-p}) R_\xi^{0 \to -p} = [\mathcal{F}].$$

where D is a connection on $E^\bullet$ which is compatible with $\bar{\partial}$.

Question

What if $\mathcal{F}$ does not have a locally free resolution on X?

To be continued.

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